Similitude and Dimensional Analysis

CE30460 - Fluid Mechanics

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Goals of Chapter

- Apply Pi Theorem
- Develop dimensionless variables for a given flow situation
- Use dimensional variables in data analysis
- Apply concepts of modeling and similitude

Basic Principles

- Dimensional Homogeneity
- In a system with several variables one can construct a series of numbers that do not have dimensions. This inherently tells you something about the scale invariance or lack thereof of a problem…..

Units and Dimensions Important in Fluids

- Primary Dimensions
  - Length (L)
  - Time (T)
  - Mass (M)
  - Temperature (θ)
- For any relationship A=B
  - Units (A)=Units (B)   Dimensional Homogeneity

Simple Example

- You are studying the pressure drop per unit length in a pipe
- Variables: Δp, D, ρ, μ, V (pressure drop per unit length, diameter, density, viscosity and velocity). If this is all we can hypothesize
  - $\Delta p = f(D, \rho, \mu, V)$
- Dimensions: $[\Delta p] = ML^{-2}T^{-2}$,
  $[D] = L$,
  $[\rho] = ML^{-3}$,
  $[\mu] = ML^{-1}T^{-1}$,
  $[V] = LT^{-1}$

Example Continued

- Dimensions: $[\Delta p]=ML^{-2}T^{-2}$,
  $[D]=L$,
  $[\rho]=ML^{-3}$,
  $[\mu]=ML^{-1}T^{-1}$,
  $[V]=LT^{-1}$

- I can combine these to form two dimensionless numbers

- $(D\Delta p)/(\rho V^2)$ and $(\rho VD)/(\mu)$

- Therefore we can say $(D\Delta p)/(\rho V^2) = F(\rho VD/\mu)$

- We don't know F, but we can look for it in data…..

Buckingham Pi Theorem

- I an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among k-r independent dimensionless products where r is the minimum number of reference dimensions required to describe the variables

- Mathematically

$$u_1 = f(u_2, u_3, \ldots, u_k)$$

can be reduced to

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

Determination of Pi terms

- List all variables that are involved in the problem
- Express each on in terms of primary dimensions (MLTθ)
- Determine required number of Pi terms (each independent) – i.e. k-r
- Select a number of repeating variables (equal to number of dimensions)
- Form pi terms by multiplying one of the nonrepeating variables by the product of repeating variables
- Repeat last step for all nonrepeating variables
- Check that all resulting Pi terms are dimensionless
- Express in final form

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

Back to Simple Example

- Applying the Methodology outlined on the last slide (let's do it on the board) we again obtain

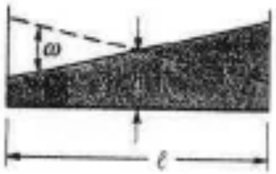
$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

By inspection

- The recipe listed before will always work
- Typically, we just do it by inspection
- All these numbers can be thought of as ratios of forces, ratios of timescales, ratios of lengthscales….

Sample Problem 1

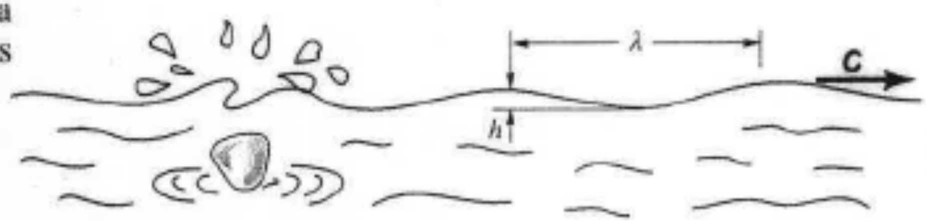
7.5 Water sloshes back and forth in a tank as shown in Fig. P7.5. The frequency of sloshing, ω, is assumed to be a function of the acceleration of gravity, g, the average depth of the water, h, and the length of the tank, ℓ. Develop a suitable set of dimensionless parameters for this problem using g and ℓ as repeating variables.



■ FIGURE P7.5

Sample Problem 2

7.9 When a small pebble is dropped into a liquid, small waves travel outward as shown in Fig. P7.9. The speed of these waves, c, is assumed to be a function of the liquid density, ρ, the wavelength, λ, the wave height, h, and the surface tension of the liquid, σ. Use h, ρ, and σ as repeating variables to determine a suitable set of pi terms that could be used to describe this problem.



■ FIGURE P7.9

Sample Problem 3

7.12 At a sudden contraction in a pipe the diameter changes from D_1 to D_2. The pressure drop, Δp, which develops across the contraction is a function of D_1 and D_2, as well as the velocity, V, in the larger pipe, and the fluid density, ρ, and viscosity, μ. Use D_1, V, and μ as repeating variables to determine a suitable set of dimensionless parameters. Why would it be incorrect to include the velocity in the smaller pipe as an additional variable?

Common dimensionless Groups in Fluids

Reynolds number	$\mathrm{Re} = \dfrac{\rho V \ell}{\mu}$
Froude number	$\mathrm{Fr} = \dfrac{V}{\sqrt{g\ell}}$
Euler number	$\mathrm{Eu} = \dfrac{p}{\rho V^2}$
Cauchy number	$\mathrm{Ca} = \dfrac{\rho V^2}{E_v}$
Mach number	$\mathrm{Ma} = \dfrac{V}{c}$
Strouhal number	$\mathrm{St} = \dfrac{\omega \ell}{V}$
Weber number	$\mathrm{We} = \dfrac{\rho V^2 \ell}{\sigma}$

Some interesting examples

- Hydraulic Jump
- Drag on a Boat
- Turbulent Flow in a Pipe

http://video.google.com/videoplay?docid=-5033243127531747249#

Data – Problems with One Pi Term

- If only one term exists then according to the Pi Theorem we can say:

$$\Pi_1 = C$$

Allows for a very nice relationship between variables and the constant can easily be fit from data….

Classical Example (Very slowly falling spherical particle – what is drag)

- Drag $\mathcal{D} = f\,(D,V,\mu)$

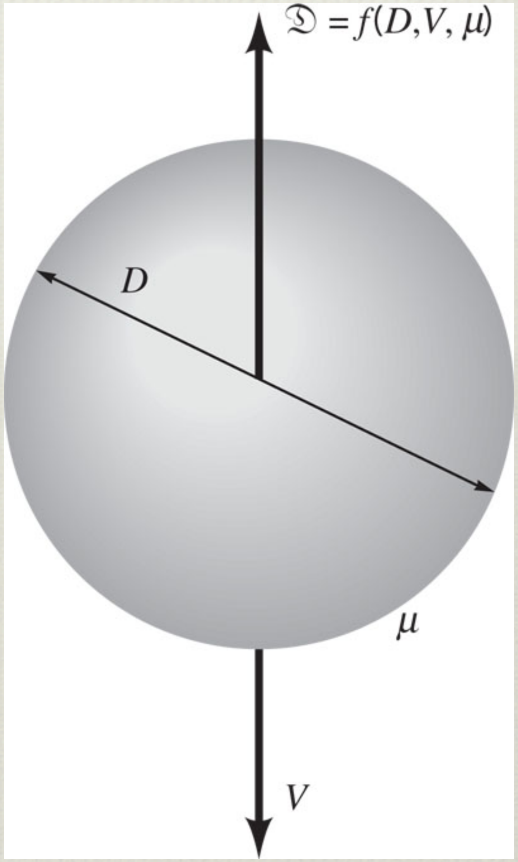
- $[\mathcal{D}] = \;\; M\,L\,T^{-2}$
  $[D] = \; L$
  $[V] = \; L\,T^{-1}$
  $[\mu] = \; M\,L^{-1}\,T^{-1}$

  4 variables
  3 dimension

- On one Pi group exists

$$\Pi_1 = \mathcal{D}/(\mu V D) = C$$

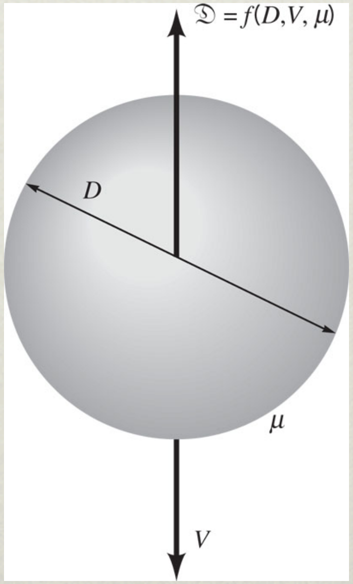
Or $\mathcal{D} = \mu\,VDC$



Sample Problem

When a sphere of diameter d=5 mm falls at a velocity of 0.1 m/s in water (viscosity=0.001 Ns/m^2) it experiences a drag of 100 N.

Can you develop a general law for drag on a sphere of arbitrary size in an arbitrary flow (for a slowing falling sphere)



Two of More Terms

- For two terms $\Pi_1=f(\Pi_2)$

$$\Pi_1=f(\Pi_2)$$

- Means when you have data you should plot Π1 against Π2 to deduce relationships or at least make best fits that can be used predicatively.

Sample Problem

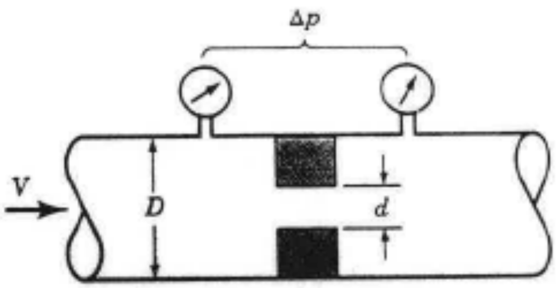
*7.23 The pressure drop across a short hollowed plug placed in a circular tube through which a liquid is flowing (see Fig. P7.23) can be expressed as

$$\Delta p = f(\rho, V, D, d)$$

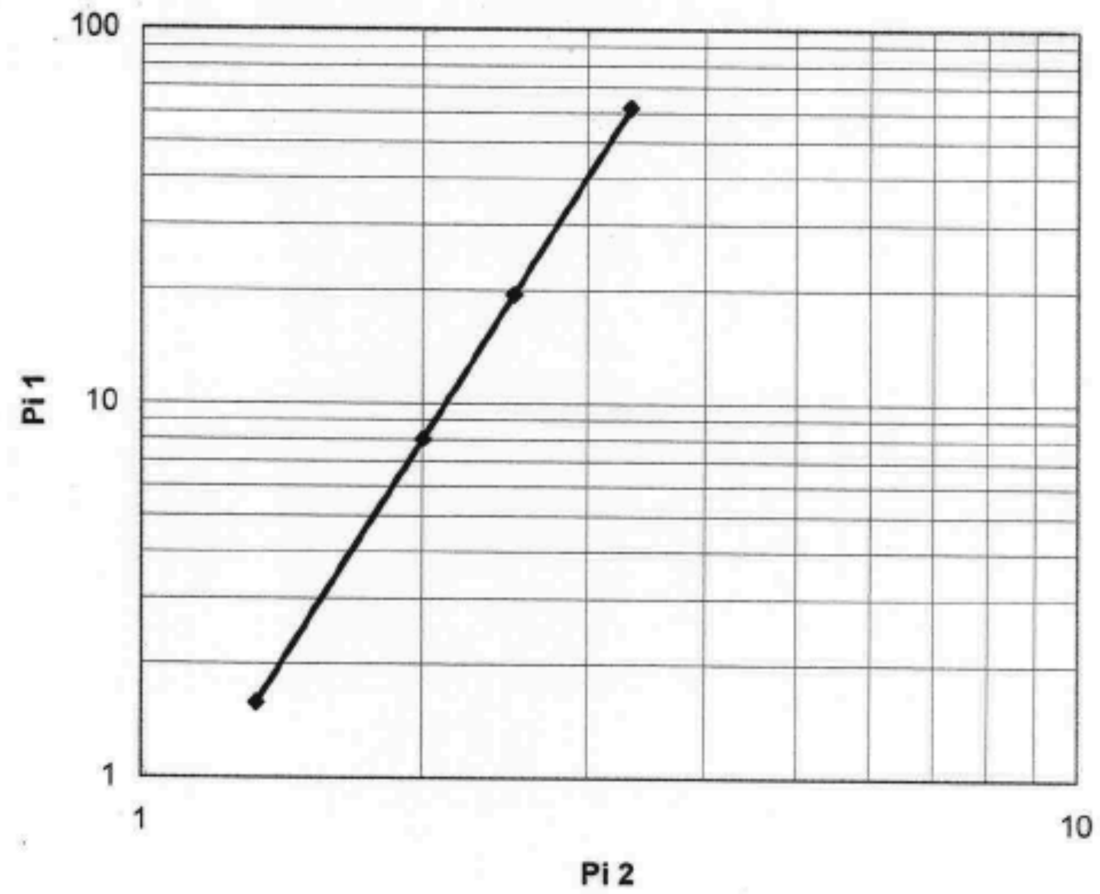
where ρ is the fluid density, and V is the mean velocity in the tube. Some experimental data obtained with $D = 0.2$ ft, $\rho = 2.0$ slugs/ft³, and $V = 2$ ft/s are given in the following table:

d (ft)	0.06	0.08	0.10	0.15
Δp (lb/ft²)	493.8	156.2	64.0	12.6

Plot the results of these tests, using suitable dimensionless parameters, on log–log graph paper. Use a standard curve-fitting technique to determine a general equation for Δp. What are the limits of applicability of the equation?

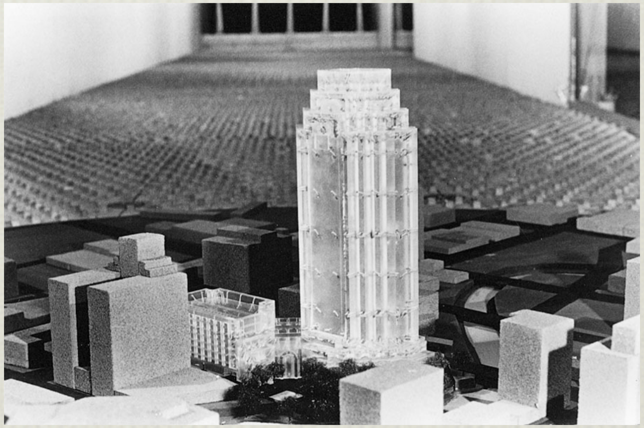


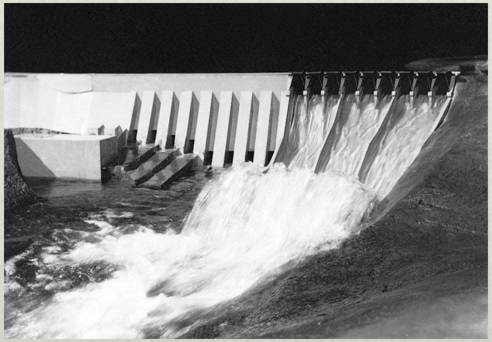
■ FIGURE P7.23


100
10
1
Pi 1
1
10
Pi 2

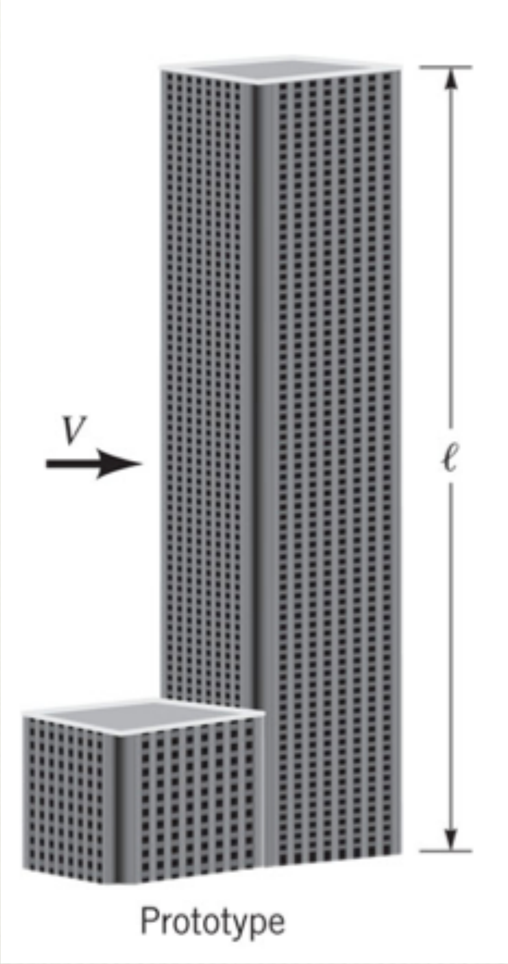
Modeling and Similitude

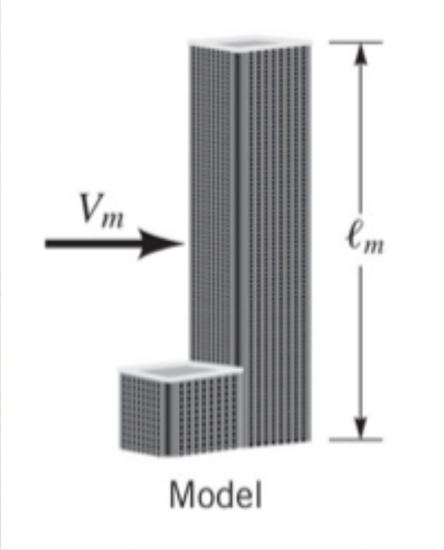
- A model is a representation of a physical system that may be used to predict the behavior of the system in some desired respect, e.g.





Is this enough?

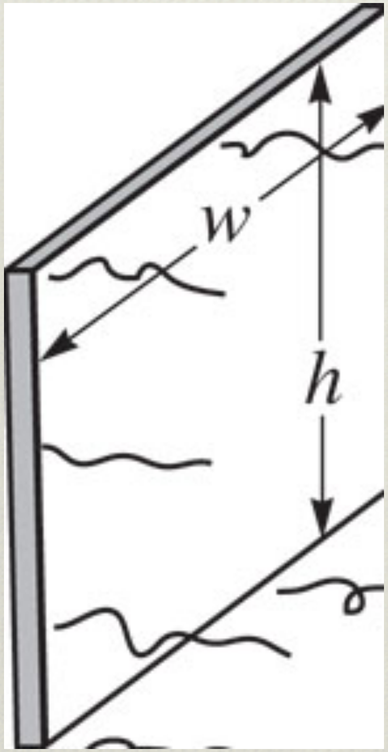




Types of Similarity

- Geometric (ratio of length scales the same)
- Kinematic (velocity structures are the same)
- Dynamic (ratio forces the same)
- The best situation is: Get all dimensionless variables (Pi groups) the same between model and prototype. Then all similarities are preserved…..
- Sometimes hard to achieve it all….

Example



- Consider flow past some plate. You can to model drag

$$\mathscr{D} = f(w,\ h,\ \mu,\ \rho,\ V)$$

- Pi theorem tells you $$\frac{\mathscr{D}}{w^2\rho V^2} = \phi\left(\frac{w}{h}, \frac{\rho V w}{\mu}\right)$$
- To ensure similarity w/h and $\rho Vw/\mu$ must be the same in model and prototype.
- If this is true $D/w^2\rho V^2$ is the same in model and prototype

3 classical examples

- Flow through Closed Conduits (Next Chapter)

- Civil Engineering Applications

  - Flow around Immersed Bodies (e.g. Buildings)
  - Flow with a Free Surface (e.g. damns, hydroelectric)

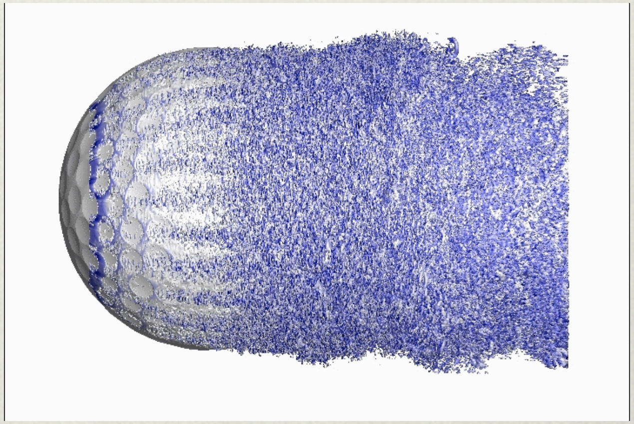
Flow around bodies

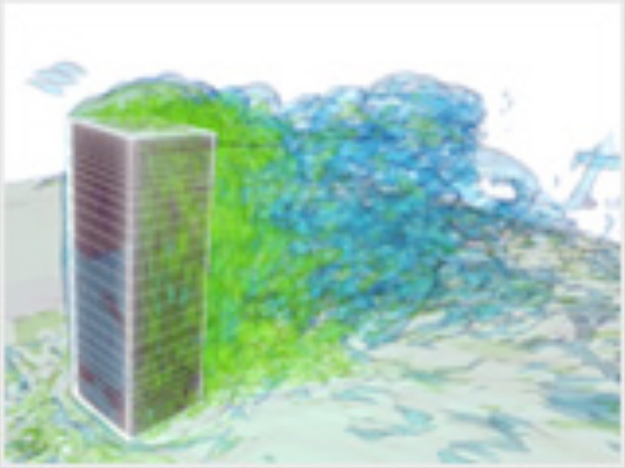
- We are often interested in drag at high Reynolds number

$$Drag = C_D \frac{1}{2} \rho V^2 l^2$$

Coefficient of drag

(what we want to measure in the lab)





Dimensionless Numbers

- C_D – is dimensionless
- Reynolds Number ($\rho V l/\mu$)
- Ratio of all length scales (l_i/l)
    - e.g. in golf ball depth of dimples over diameter of ball
    - In building width/height, window length/height, etc.
- Using Pi Theorem

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 l^2} = \phi\left(\frac{l_i}{l}, \frac{\rho V l}{\mu}\right)$$

As long as the Reynolds numbers and length scale ratios are the same in the model as the actual case (prototype) C_D will the same

Relationship between Prototype and Model

- We can now calculate the drag on the prototype from measurements on the model.

$$\frac{D_M}{\frac{1}{2}\rho_M V_M{}^2 l_M{}^2} = \frac{D_P}{\frac{1}{2}\rho_P V_P{}^2 l_P{}^2}$$

$$D_P = \frac{\rho_P V_P{}^2 l_P{}^2}{\rho_M V_M{}^2 l_M{}^2} D_M$$

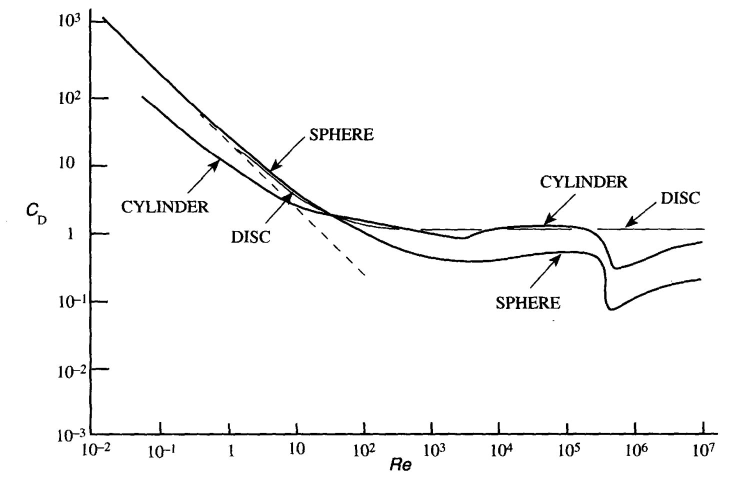
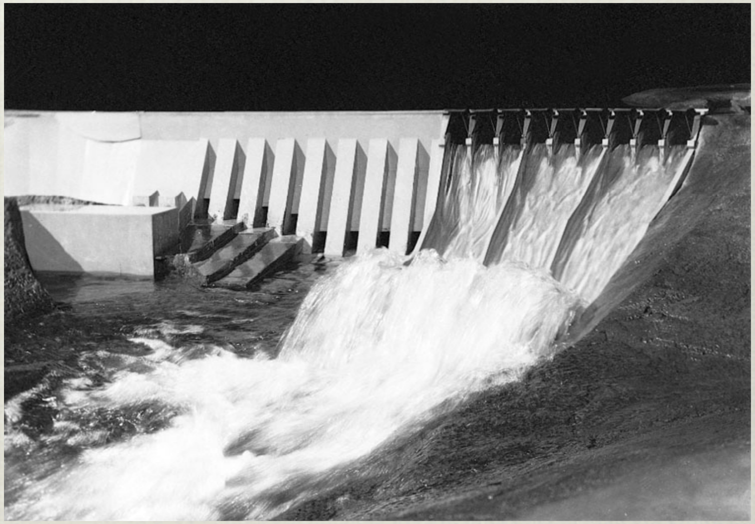


Figure 7.13 Log–log plot of drag coefficient C_D as a function of Reynolds Number *Re* for spheres, transverse cylinders, and face-on discs. The broken straight line represents Stokes's law.

Flow with a Free Surface

- Gravity Driven Flows (i.e. water flows down a hill)

- As long as all other dimensionless numbers are matched between model and prototype Froude Number only matters (tough to do - e.g. Reynolds number – make large enough in both cases to not be important)



$$Fr_M = Fr_M \quad \Rightarrow \quad \frac{V_M}{\sqrt{g l_M}} = \frac{V_P}{\sqrt{g l_P}}$$

$$V_P = \frac{\sqrt{l_P}}{\sqrt{l_M}} V_M$$

7.35 The pressure rise, Δp, across a blast wave, as shown in Fig. P7.35 is assumed to be a function of the amount of energy released in the explosion, E, the air density, ρ, the speed of sound, c, and the distance from the blast, d. (a) Put this relationship in dimensionless form. (b) Consider two blasts: the prototype blast with energy release E and a model blast with 1/1000th the energy release ($E_m = 0.001\ E$). At what distance from the model blast will the pressure rise be the same as that at a distance of 1 mile from the prototype blast?

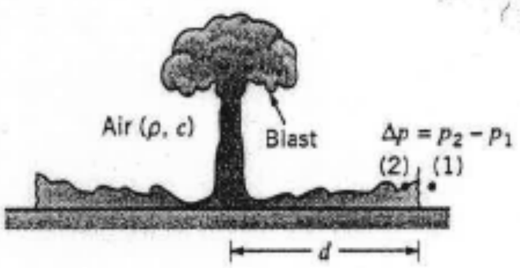


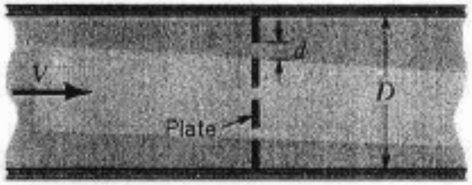
FIGURE P7.35

Sample Problems

7.41 As shown in Fig. P7.41, a thin, flat plate containing a series of holes is to be placed in a pipe to filter out any particles in the liquid flowing through the pipe. There is some concern about the large pressure drop that may develop across the plate, and it is proposed to study this problem with a geometrically similar model. The following data apply.

(a) Assuming that the pressure drop, Δp, depends on the variables listed, use dimensional analysis to develop a suitable set of dimensionless parameters for this problem. (b) Determine values for the model indicated in the list with a question mark. What will be the pressure drop scale, $\Delta p_m/\Delta p$?

Prototype	Model
d—hole diameter = 1.0 mm	d = ?
D—pipe diameter = 50 mm	D = 10 mm
μ—viscosity = 0.002 N·s/m^2	μ = 0.002 N·s/m^2
ρ—density = 1000 kg/m^3	ρ = 1000 kg/m^3
V—velocity = 0.1 m/s to 2 m/s	V = ?



■ FIGURE P7.41

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Allows for a very nice relationship between variables and the constant can easily be fit from data….