Chapter 6 Differential Analysis of Flow

CE30460 - Fluid Mechanics

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Chapter Goals

- Kinematics of given flow field
- Continuous Continuity Equation
- Navier-Stokes and Specific Solutions
- Concepts of Potential Flow

Kinematics

- Recall Material Derivative (Lagrangian vs Eulerian)

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}$$

- Acceleration of a fluid element

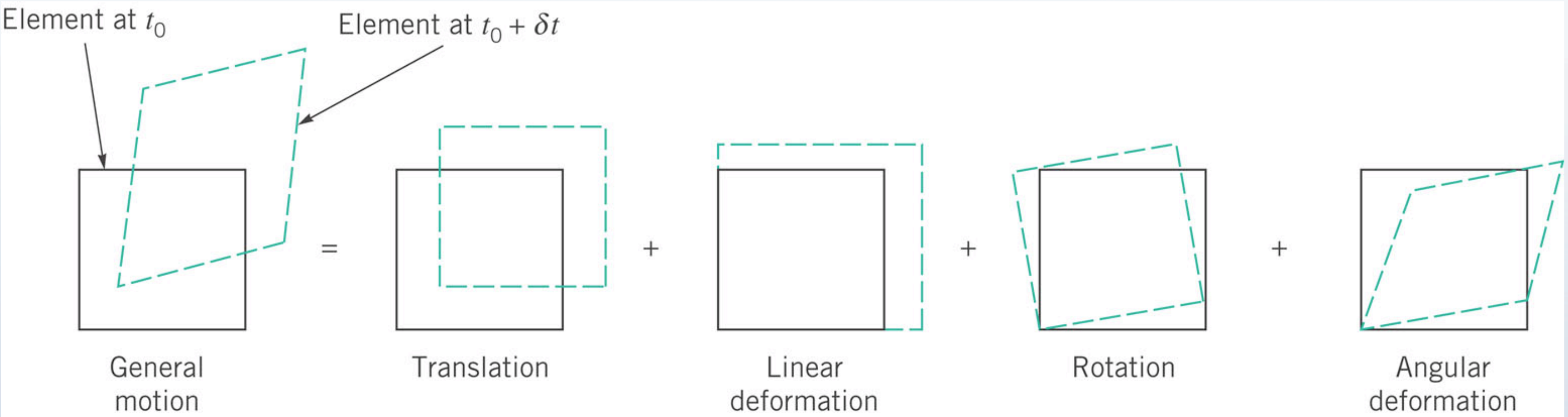
$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

Types of Motion

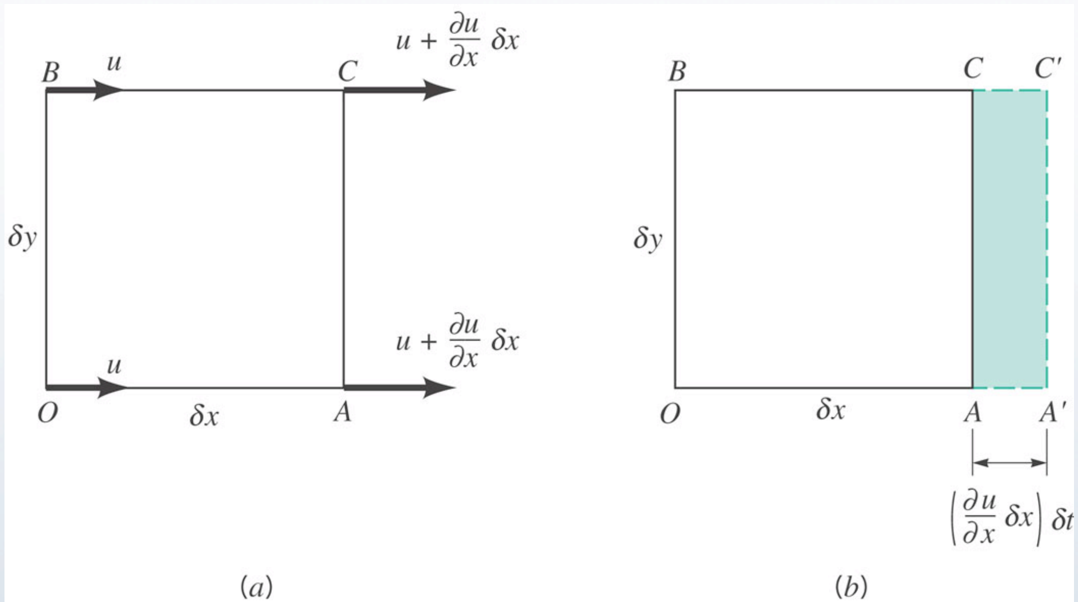
- Translation
- Linear Deformation
- Rotation
- Angular Deformation

  - Draw each of these for a rectangular initial element....

Types of Motion



Linear Motion and Deformation



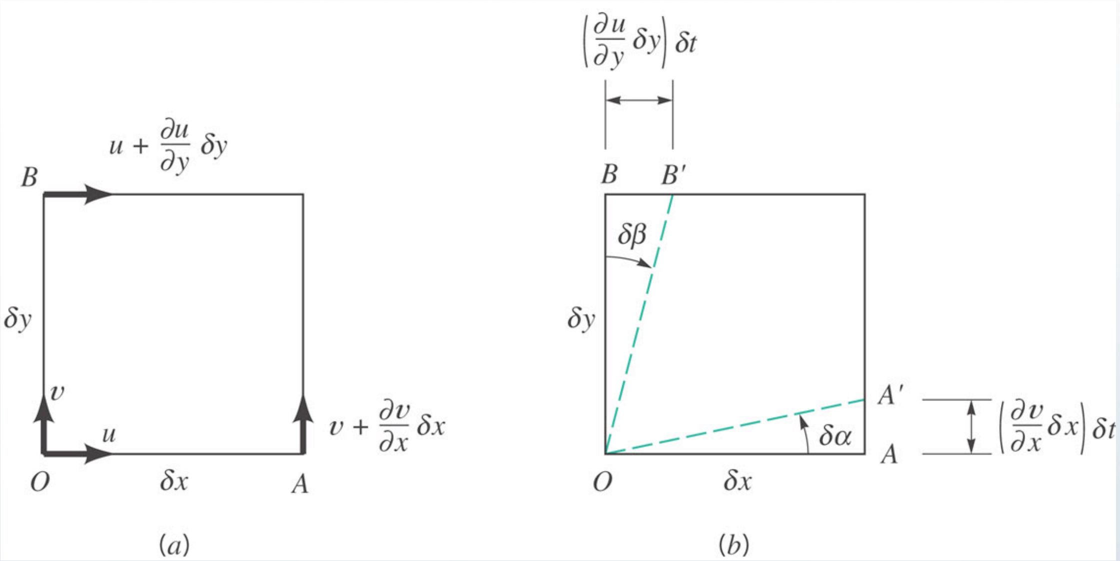
$$\frac{1}{\delta \forall}\frac{d(\delta \forall)}{dt} = \lim_{\delta t \to 0}\left[\frac{(\partial u/\partial x)\,\delta t}{\delta t}\right] = \frac{\partial u}{\partial x}$$

In 3 dimensions

- Volumetric Dilatation Rate

$$\frac{1}{\delta \forall} \frac{d(\delta \forall)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \boldsymbol{\nabla} \cdot \mathbf{V}$$

Angular Motion and Deformation



$$\tan\partial\alpha \approx \alpha = \frac{(\partial v/\partial x)\partial x\partial t}{\partial x} = \frac{\partial v}{\partial x}\partial t \qquad \tan\partial\beta \approx \beta = \frac{(\partial u/\partial y)\partial y\partial t}{\partial y} = \frac{\partial u}{\partial y}\partial t$$

$$\omega_{OA} = \lim_{\partial t\to 0}\frac{\partial\alpha}{\partial t} = \frac{\partial v}{\partial x} \qquad \omega_{OB} = \lim_{\partial t\to 0}\frac{\partial\beta}{\partial t} = \frac{\partial u}{\partial y}$$

Vorticity (Rotation)

- Counterclockwise rotation is positive (z component is component out of the page). Others, in x-y also exist

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

- vorticity (zero => irrotational)

$$\boldsymbol{\zeta} = 2\,\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{V}$$

- Related, rate of shearing strain

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Sample Problem 1

6.5 A one-dimensional flow is described by the velocity field

$$u = ay + by^2$$

$$v = w = 0$$

where a and b are constants. Is the flow irrotational? For what combination of constants (if any) will the rate of angular deformation as given by Eq. 6.18 be zero?

Sample Problem 2

6.7 An incompressible viscous fluid is placed between two large parallel plates as shown in Fig. P6.7. The bottom plate is fixed and the upper plate moves with a constant velocity, U. For these conditions the velocity distribution between the plates is linear, and can be expressed as

$$u = U\frac{y}{b}$$

Determine: **(a)** the volumetric dilatation rate, **(b)** the rotation vector, **(c)** the vorticity, and **(d)** the rate of angular deformation.

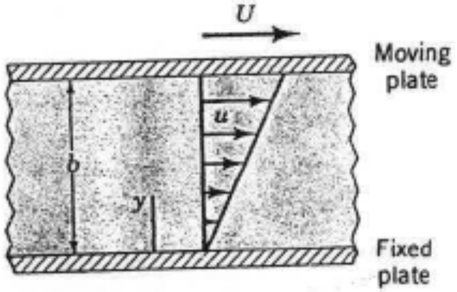
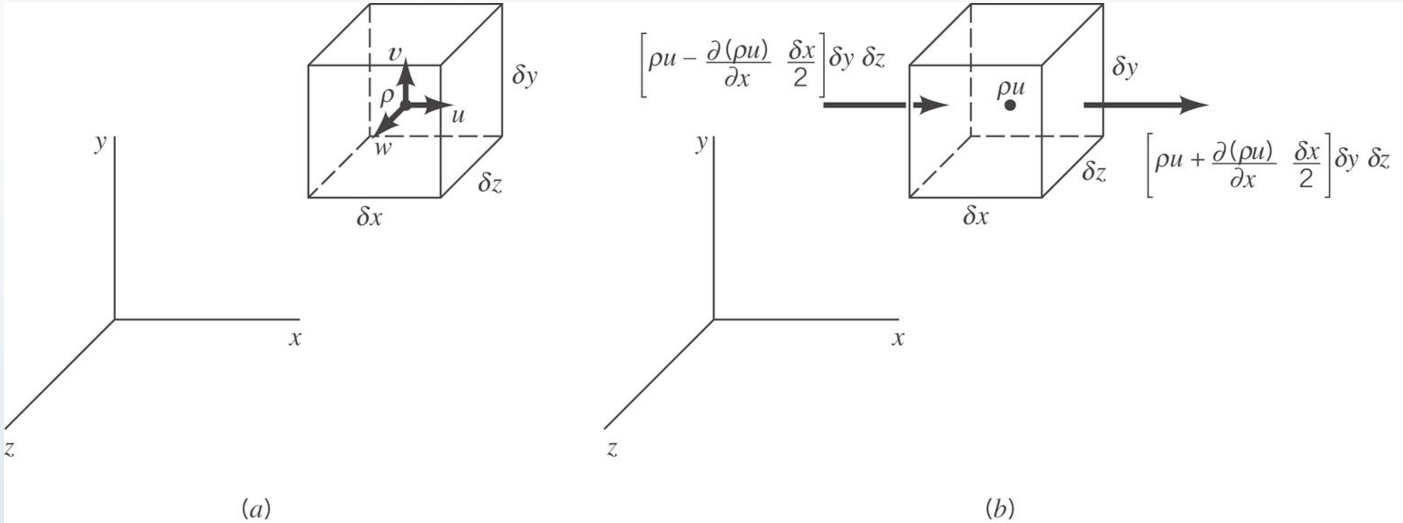


FIGURE P6.7

Conservation of Mass



$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \sum \rho_{out} A_{out} V_{out} - \sum \rho_{in} A_{in} V_{in} = 0$$

But now in the limit of zero volume (to obtain differential form)

Forms of Continuity Equation

- General Form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

- Steady

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

- Incompressible

$$\nabla \cdot \mathbf{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical Polar Coordinates

- General Form

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

- Steady

$$\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

- Incompressible

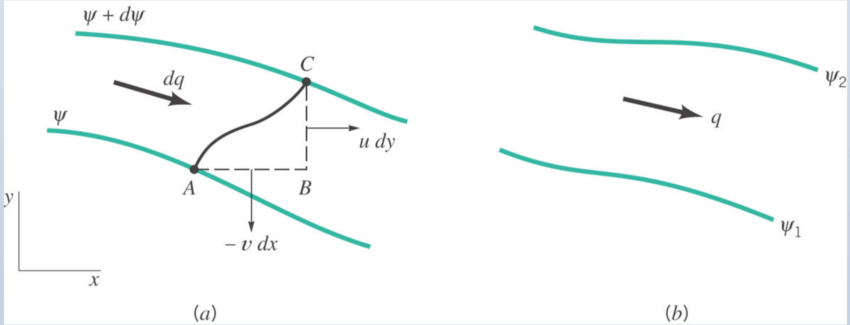
$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Streamfunction

- For incompressible, plane two dimensional flow we can define a streamfunction psi, such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

- Quantifies the flow rate between two streamlines (lines of constant psi)



$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

In cylindrical polar coordinates

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}=0$$

$$v_r=\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta=-\frac{\partial \psi}{\partial r}$$

Sample Problem 1

6.11 The radial velocity component in an incompressible, two-dimensional flow field ($v_z = 0$) is

$$v_r = 2r + 3r^2 \sin \theta$$

Determine the corresponding tangential velocity component, v_θ, required to satisfy conservation of mass.

Sample Problem 2

6.21 The velocity components in an imcompressible, two-dimensional flow field are given by the equations.

$$u = x^2$$
$$v = -2xy + x$$

Determine, if possible, the corresponding stream function.