Homework 4

(1) $x = 100$

$M = 0.1$ kg/m

For a pulse $\Rightarrow$ $C(x,t) = \frac{1}{(4\pi D t)} e^{\frac{-(x-vt)^2}{4Dt}}$

Guess $v \approx 0.2$ m/s (peak arrives around 500 s)

Guess $D \approx \frac{1}{4\pi t_{peak} C_{peak}^2} \approx 0.05$ m^2/s

(from $C_{peak} = \frac{1}{\sqrt{4\pi D t}}$)

If you use these guesses you get an excellent match to data

$\therefore$ Take $v = 0.2$ m/s $D = 0.05$ m^2/s

Now for a steady state plume

$$C = C_o e^{\left(\frac{v - \sqrt{v^2 + 4D\gamma}}{2D}\right)x}$$

Rearranging $\gamma = \frac{\left[v - \frac{2D}{x} \ln\left(\frac{C}{C_o}\right)\right]^2 - v^2}{4D}$

We know $x = 100$, $\frac{C}{C_o} = 0.1$, $D = 0.05$, $v = 0.2$

$\therefore$ $\gamma = 0.0046$ s^{-1}

If we only have advection

$$C = C_0 e^{-\frac{\gamma x}{v}}$$

$$\Rightarrow \gamma = -\frac{v}{x} \ln\left(\frac{C}{C_0}\right) \approx 0.0046 \ s^{-1}$$

$\downarrow$

practically the same

If we only have dispersion

$$C = C_0 e^{-\sqrt{\frac{\gamma}{D}}\, x}$$

$$\Rightarrow \gamma = D\left[\frac{1}{x}\ln\left(\frac{C}{C_0}\right)\right]^2 = \underbrace{2.7 \times 10^{-5} \ s^{-1}}_{\text{much smaller}}$$

$\therefore$ In this system we can neglect dispersion

(ii) $$C = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-vt)^2}{4Dt}} e^{-\gamma t}$$

$$\left.\begin{aligned} x &= 100 \ m \\ v &= 0.2 \ m/s \\ D &= 0.05 \ m^2/s \\ \gamma &= 0.0046 \ s^{-1} \end{aligned}\right\}$$

Plot - you will see that concentrations, particularly at late times are much less (by a factor of $e^{-\gamma t}$) than conservative case

② Governing Equations

$$\frac{\delta C_A}{\delta t} = 2\frac{\delta^2 C_A}{\delta x^2} - r \qquad C_A(t=0) = \delta(x+10) \qquad (1)$$

$$\frac{\delta C_B}{\delta t} = 2\frac{\delta^2 C_B}{\delta x^2} - r \qquad C_B(t=0) = \delta(x-10) \qquad (2)$$

$$\frac{\delta C_c}{\delta t} = 2\frac{\delta^2 C_c}{\delta x^2} + r \qquad C_c(t=0) = 0 \qquad (3)$$

Define $u_A = C_A + C_c$

$u_B = C_B + C_c$

From (1)-(3) $\frac{\delta u_A}{\delta t} = 2\frac{\delta^2 u_A}{\delta x^2} \quad u_A(t=0) = \delta(x+10)$

$$\frac{\delta u_B}{\delta t} = 2\frac{\delta^2 u_B}{\delta x^2} \qquad u_B(t=0) = \delta(x-10)$$

$$\therefore u_A = \frac{1}{\sqrt{8\pi t}} e^{-\frac{(x+10)^2}{8t}}$$

$$u_B = \frac{1}{\sqrt{8\pi t}} e^{-\frac{(x-10)^2}{8t}}$$

Now $c_A c_B = 0$ (they cannot coexist)

$\therefore$ If $u_A = c_A + c_C$

$u_B = c_B + c_C$

If $u_A > u_B \Rightarrow c_A > c_B \Rightarrow c_B = 0 \Rightarrow c_C = u_B$

$u_B > u_A \Rightarrow c_B > c_A \Rightarrow c_A = 0 \Rightarrow c_C = u_A$

$\therefore \quad c_C = \min(u_A, u_B)$

From symmetry of problem you should see

$u_A > u_B$ for $x < 0$

$u_B > u_A$ for $x > 0$

$\therefore \quad c_C = u_B$ for $x < 0$

u_A for $x > 0$

$$M_C = \int_{-\infty}^{\infty} c_C \, dx = \int_{-\infty}^{0} u_B \, dx + \int_{0}^{\infty} u_A \, dx$$

$$= 2\int_{0}^{\infty} u_A \, dx$$ (again from symmetry)

$$\therefore M_c = 2\int_0^{\infty} \frac{1}{\sqrt{8\pi t}} e^{-\frac{(x+10)^2}{8t}} \, dx$$

$$= \mathrm{erfc}\left(\frac{\sqrt{5}}{\sqrt{2t}}\right) \text{ from Mathematica}$$

which you can now calculate and plot at desired times

(ii) The only thing that changes is that everything moves to the right at velocity v. Nothing else.

(iii) Our solutions for u_A, u_B change and so also M_c

$$M_c = \mathrm{erfc}\left(\frac{5}{\sqrt{Dt}}\right)$$

D=4 instead of 2

(iv) At early times the same

At late time u_A, u_B = constant $\Rightarrow$ Domain filled evenly with C; no A+B survive.