Homework 1

(i) $$\frac{\delta^2 C}{\delta t^2} = v^2 \frac{\delta^2 C}{\delta x^2} \qquad C(t=0) = \delta(x)$$

$$\left.\frac{\delta C}{\delta t}\right|_{t=0} = 0$$

Fourier Transform

$$\frac{\delta^2 \hat{C}}{\delta t^2} = -v^2 k^2 \hat{C} \qquad \hat{C}(t=0) = 1$$

$$\frac{\delta \hat{C}}{\delta t}(t=0) = 0$$

What function when you differentiate it twice, gives you the same function, but negative (or Mathematica) ~ sin and cos

solution $\Rightarrow$ $\hat{C} = A \sin(vkt) + B \cos(kvt)$

Check $$\frac{\delta \hat{C}}{\delta t} = Avk \cos(kvt) - Bvk \sin(kvt)$$

$$\frac{\delta^2 \hat{C}}{\delta t} = -A v^2 k^2 \sin(vkt) - B v^2 k^2 \cos(kvt) = -v^2 k^2 \hat{C}$$

↓ GOOD

A, B constants that depend on initial conditions

$$\left.\begin{aligned} \hat{C}(t=0) = 1 &\Rightarrow B = 1 \\ \frac{\delta \hat{C}}{\delta t}(t=0) = 0 &\Rightarrow A = 0 \end{aligned}\right\} \hat{C} = \sin(vkt)$$

Inverse Fourier Transform (from Mathematica or tables) $\Rightarrow$ $C = \frac{1}{2}\delta(x - vt) + \frac{1}{2}\delta(x + vt)$

↳ half mass goes left and half right at speed V as pulses

(2) $$\frac{\delta C}{\delta t} = \frac{\delta^2 C}{\delta x^2} + \frac{\delta^2 C}{\delta y^2} \qquad C(x,y,t=0) = \delta(x)\delta(y)$$

Fourier transform in x-direction

$$\frac{\delta \hat{C}}{\delta t} = -k^2 \hat{C} + \frac{\delta^2 \hat{C}}{\delta y^2} \qquad \hat{C}(k,y,t=0) = \delta(y)$$

Fourier transform this equation in y-direction

$$\frac{\delta \tilde{\hat{C}}}{\delta t} = -k^2 \tilde{\hat{C}} - l^2 \tilde{\hat{C}} \qquad \tilde{\hat{C}}(k,l,t=0) = 1$$

Solve O.D.E. $$\frac{\delta \tilde{\hat{C}}}{\delta t} = -(k^2+l^2)\tilde{\hat{C}}$$

$$\tilde{\hat{C}} = A e^{-(k^2+l^2)t}$$

$$= e^{-(k^2+l^2)t}$$ because $\tilde{\hat{C}}(t=0) = 1$ $(A=1)$

Now $$\tilde{\hat{C}} = e^{-k^2 t} e^{-y^2 t}$$

Inverse Fourier Transform $(l \to y)$ $\Rightarrow$ $$\hat{C} = e^{-k^2 t} \frac{1}{\sqrt{4\pi D t}} e^{-y^2/4Dt}$$

Inverse Fourier Transform $(k \to x)$ $\Rightarrow$ $$C = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/4Dt} \frac{1}{\sqrt{4\pi D t}} e^{-y^2/4Dt}$$

See notes or Mathematica

$$= \frac{1}{4\pi D t} e^{-\frac{(x^2+y^2)}{4Dt}}$$

For 3-d $\Rightarrow$ $$\frac{1}{(4\pi D t)^{3/2}} e^{-\frac{(x^2+y^2+z^2)}{4Dt}}$$

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%HW1 - FT

%THe key to this problem is to recognize that the two functions given are
%inverse functions. That is the given f(k) is the Fourier transform of f(x)
%and vice versa so the comparison is easy.

clear
clc
close all

%Define our Domains in x and k

x=-10:0.01:10;  %this is domain that goes from x=-10 to 10 in increments of 0.01;
k=-10:0.01:10;  %same for Fourier domain

%Define the function we want to Fourier Transform
fx=1/sqrt(4*pi)*exp(-x.^2/4);

%Define the function we want to Inverse Fourier Transform
fk=exp(-k.^2);


%Now let's implement the Fourier Transform of fx and calculate it at
%several points in k space. Recognize that the integral can be approximated
%by a sum \int f(x) dx = sum f(x) \Delta x

%Loop over each k point

dx=x(2)-x(1)      %spatial step

for jj=1:length(k)

  Fk(jj)=sum(fx.*exp(i*k(jj)*x))*dx;

end

figure(1)
hold on
plot(k,fk,'.')
plot(k,Fk,'r')
xlabel('k')
ylabel('f(k)')
legend('Analytical','Numerical')
title('Fourier Trasform of f(x)')
%Now let's implement the Invserse Fourier Transform of fk and calculate it at
%several points in x space. Recognize that the integral can be approximated
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%by a sum \int f(k) dk = sum f(k) \Delta k

%Loop over each k point

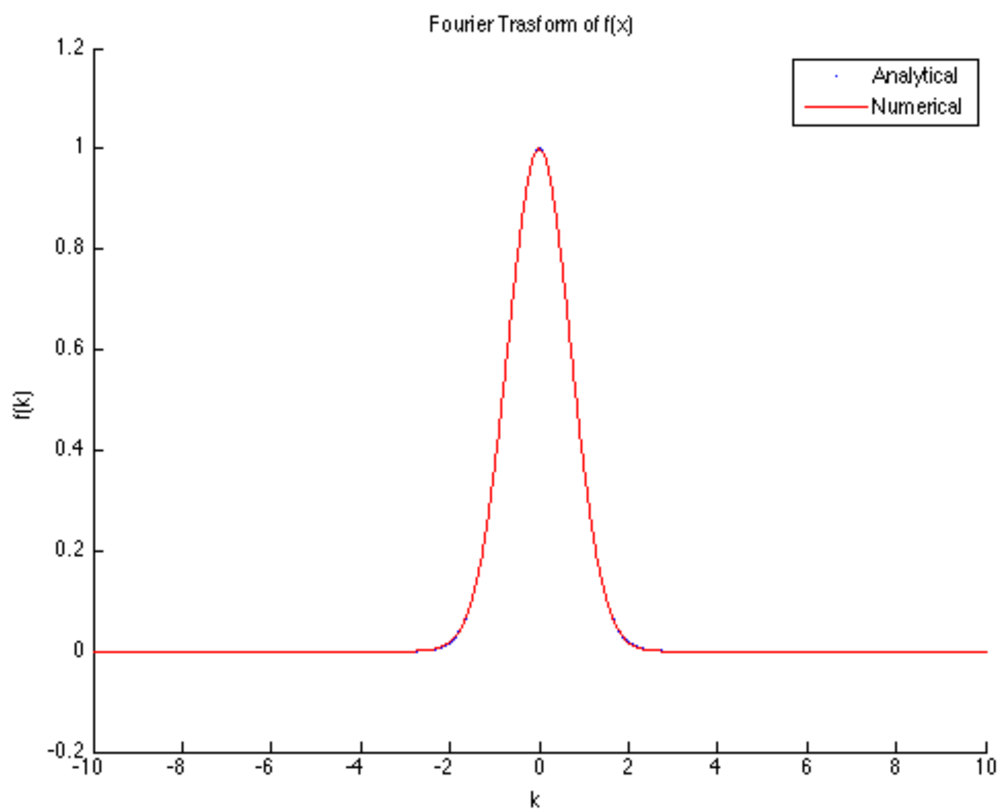
dk=k(2)-k(1)      %spatial step

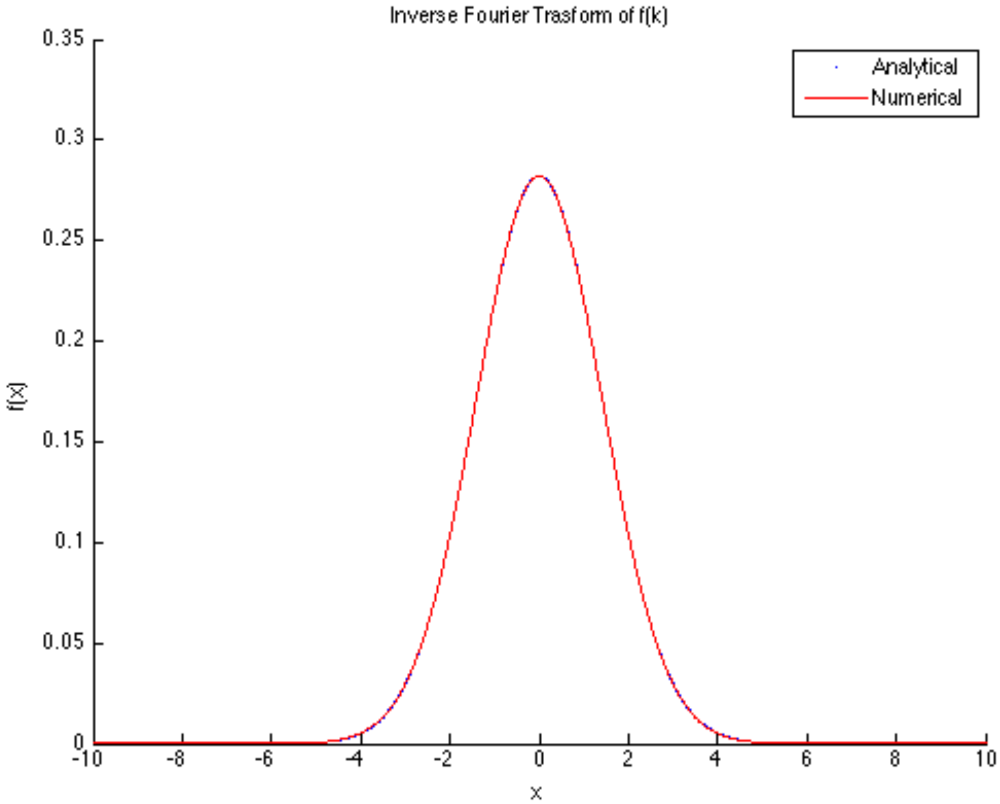
for jj=1:length(x)

   Fx(jj)=1/2/pi*sum(fk.*exp(-i*k*x(jj)))*dk;

end

figure(2)
hold on
plot(x,fx,'.')
plot(x,Fx,'r')
xlabel('x')
ylabel('f(x)')
legend('Analytical','Numerical')
title('Inverse Fourier Trasform of f(k)')
```




Inverse Fourier Trasform of f(k)
Analytical
Numerical
f(x)
x
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-10
-8
-6
-4
-2
0
2
4
6
8
10